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Derivatives for Solving Optimization with Constraints In Business And Economics

Nurul Yaqin

Lecturer at STMIK Bahrul 'Ulum, Jombang
Indonesia

Email address: nurul_yaqin@yahoo.com

Abstract. Mathematics is a tool used by other scientific disciplines in order of solving the problem can be obtained in quantitative results. The optimum problem is an interesting one type of problem that will be the object of our discussion in this paper. The main idea in this paper shows how easy substitution method and Lagrange Multiplier Methods in solving problems of optimum with constraints. In this paper the author will show you a little bit about how easy and smooth optimum mathematical problem solving using formulas derivative / differential for business and economics. With this simple example of the problems that the authors hope to provide an overview to practitioners on how business and economic potential-kensep math concepts can help them. And also note the importance of simplification of a problem with using mathematical modeling approach, so that mathematical formulas can help them in solving the problem with a good or optimal results. And finally it would seem that the business and the nature of mathematical economics.

Key-Words: *Substitution Method, Lagrange Multiplier*

1. Background

In life, we have always focused on solving business problems efficiently. This is in order to find the best solution to solve the economic problems and business. This thing we need to do given the limited available resources (in the form of funds, time, energy and natural resources). Given limited resources, this is always an obstacle or in any constraint we run the calculation process to achieve optimal results or the best.

On the other hand with the rapid development of the mathematical world, a lot of economic problems and emerging business that can be solved with theories or mathematical formulas, including problems to achieve efficiency with various constraints as mentioned above. In mathematics, the problems to achieve this efficiency is known as Optimization (Dumairy, 1998).

Problem. And specifically for optimization problems that are full of obstacles we call Constrained Optimization (optimization with constraints).

Constraints of optimization which can be solved by using the theories and formulas derivatives that exist in mathematics (Hillier, et.al.,1994). For example, the constraints problem which can be solved with a derivative formulas are: A group of customers who are facing the utility function U = 3Q1Q2 + 3Q2 with Q1 (the first product) and Q2 (second product). 10Q2 20Q1 + = 100 is constraint the customer budget that must be considered in efforts to achieve maximum utility. Due to budget limitations (ie: 100) that it is unlikely that the group has the customer must purchase the first product (price: 20) and a second product (price: 10) are not limited to (entirely) in order to achieve utility (maximum satisfaction).

With the example above, we can understand that, constraints of optimization which is a realistic problem that will always we encounter in everyday life, both in the production process and in the process to enjoy the results of the production.

2. Problem Formula

With the inclusion of constraints in the optimization problem which will be discussed later, derivative formulas to find the maximum or minimum value in mathematics there should be engineered so that these constraints can be included in the calculation of optimization.

Before use derivative formulas to solve them, and see obstacles still simple (there are only two variables Q_1 and Q_2), should also be held on the reformulation of the problem along with the obstacles by using the methods:
1 Method of substitution
2 lagrange multiplier method

3. Theoretical Frameworks

To obtain optimum results, the mathematical calculation requirements need to be met as follows:
1 Terms of need, which must satisfy the first order derivative.
2 Terms sufficient, ie the second order derivatives must satisfy
Derivative formulas:

$f(x) = ax^n + bx + c$ example : $f(x) = 2x^3 + 3x + 6$

$f'(x) = anx^{n-1} + b$ $f(x) = 6x^2 + 3$

$f''(x) = 12x$

$y = ax_1^n x_2 + bx_1x_2 + c$ example : $y = f(x_1, x_2) = 2x_1^3 x_2 + 3x_1x_2 + 5$

$y_1 = dy/dx_1 = anx_1^{n-1}x_2 + bx_2$ $f_1 = df/dx_1 = 6x_1^2 x_2 + 3x_2$

$y_2 = dy/dx_2 = ax_1^n + bx_1$ $f_2 = df/dx_2 = 2x_1^3 + 3x_1$

$f_{11} = d^2f/dx_1^2 = 12x_1x_2$

$f_{22} = d^2f/dx_2^2 = 0$

$f_{12} = f_{21} = d^2f/dx_1dx_2 = d^2f/dx_2dx_1$

$= 6x_1^2 + 3$

The requirements for optimum functioning: $Z = f(x_1, x_2, x_3, \ldots, x_n)$

Table 1. The Requirement of Z Optimum Function

Requirement	maximum	minimum
First Order	$f_1 = f_2 = f_3 = \ldots = f_n = 0$	$f_1 = f_2 = f_3 = \ldots = f_n = 0$
Second Order	$\lvert H_1 \rvert < 0; \lvert H_2 \rvert > 0;$ $\lvert H_3 \rvert < 0; \ldots.$ or d^2z definite negative	$\lvert H_1 \rvert; \lvert H_2 \rvert; \lvert H_n \rvert > 0$ or d^2z definite positive

The requirements for optimum functioning $Z = f(x_1, x_2, \ldots, x_n)$, that satisfies constraints (constraints) g (x1, x2, ..., xn) = c with

Table 2. The Requirement of Z Optimum Function with Constraints

Requirement	maximum	minimum
First Order	$Z_{\ldots} = Z_1 = Z_2 = \ldots = Z_n = 0$ or $dz = 0$ bound on $g = c$	$Z_{\ldots} = Z_1 = Z_2 = \ldots = Z_n = 0$ or $dz = 0$ bound on $g = c$
Second Order	$\lvert \overline{H}_2 \rvert > 0; \lvert \overline{H}_3 \rvert < 0; \lvert \overline{H}_4 \rvert > 0$ or d^2z definite negative, bound on $dg = 0$	$\lvert \overline{H}_2 \rvert, \lvert \overline{H}_3 \rvert, \ldots, \lvert \overline{H}_n \rvert < 0$ or d^2z definite positive, bound on $dg = 0$

4. Substitution Method

As mentioned above, to solve the problem optimum particularly simple, can be done with subtsitution method. Discussion we begin by observing the utility function U = 3Q1Q2 + 3Q2 faced by a group of customers by adding a budget constraint (budget) that is owned by the customer. If the customer provides a budget of 100 to buy both types of goods Q1 and Q2, while the prevailing price for Q1 and Q2 respectively are 20 and 10, the linear equation for the budget is 10Q2 20Q1 + = 100.

To find the optimal values of the above simple problem we can still use the following techniques:

$U = 3Q_1Q_2 + 3Q_2$

$20Q_1 + 10Q_2 = 100$ $2Q_1 + Q_2 = 10$

$Q_2 = 10 - 2Q_1$

Now substitute equation (2) into equation (1) to obtain

$U = 3Q_1 (10 – 2Q_1) + 3 (10 – 2Q_1)$
$= 30Q_1 – 6Q_1^2 + 30 – 6Q_1$
$= – 6Q_1 + 24Q_1 + 30$

From $U = – 6Q_1^2 + 24Q_1 + 30$, the first order derivatives can be determined U_1 $dU/dQ_1 = – 12Q_1 + 24$
In order to obtain the maximum U Terms of necessity is $dU/dQ_1 = 0$.

$– 12Q_1 + 24 = 0$
$Q_1 = 2$

To find the value of Q2 is substituting (3) into equation (2) as follows:
$Q_2 = 10 – 2 (2)$
$= 6$

Furthermore, substituting the values of (3) and (4) to equation (1)

$U = 3 (2) (6) + 3 (6)$
$= 54$

So, with the values Q1 = 2 and Q 2 = 6 obtained the optimal value with constraint

U = 54

5. Lagrange Multiplier Method

Another way that can be used to solve the optimum problem with a more complex constraint functions is Multiplier Method (indefinite) Lagrange. The essence of this method is to transform a functional form such that the first order condition of problem free extreme can still be used to solve a problem extreme (optimization) with constraints (Chiang, 1983).

To know that the Lagrange multiplier method is more general than the technique or method of settlement of the above, it is better if we use this method to discuss or resolve the same problem (as above).

As has been shown (discussed) above, the process to maximize the value of the utility function U=U + 3Q2 3Q1Q2 bound to constraint 10Q2 20Q1 + = 100 is a optimization problem which has been successfully completed by using a technique or method that is relatively simple. But to know and prove the general nature of the multiplier method Lagrange, the above problem let us try to solve by using the Lagrange Multiplier method.

The first step that needs to be done is to establish a lagrange function which is a modified version of an objective function U (as above) which coupled with constraints in the form of as-follows:

$$Z = 3Q_1Q_2 + 3Q_2 + \lambda(100 - 20Q_1 - 10Q_2)$$

λ (lambda) is the symbol of which is a number that has not been determined, which is referred to as a multiplier (indeterminate) lagrange. In this case, if constraint can be met, regardless of the value, then the last term in the equation above will disappear so that the function U will be equal to the function of Z. Thus we will be able to perform the optimization without being disturbed again by obstacles.

However, the problem is how to manipulate the lagrange function so its constraint missing from the equation the function Z (due to have met the demands its constraint). In other words, we can run the optimization process of the functions U free as compensation for the optimization with constraint connection with the fulfillment of these constraints.

Techniques to obtain the expected results is to assume that enough is as an additional variable in the function Z, ie Z = z (, Q1, Q2). Thus the first order condition (necessary condition) for free extremes will consist of sets of simultaneous equations, as follows.

$$Z_\lambda \equiv dz/d\lambda = 100 - 20Q_1 - 10Q_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$Z_1 \equiv dz/dQ_1 = 3Q_2 - 20\lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$Z_2 \equiv dz/dQ_2 = 3Q_1 + 3 - 10\lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

If equation (1), (2) and (3) be solved simultaneously obtained values

$$Q_1 = 2, Q_2 = 6 \quad and\ \lambda = 9/10$$

The values obtained with the above calculations as follows:

$$100 - 20Q_1 - 10Q_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$3Q_2 - 10\lambda = 0 \quad \Leftrightarrow \quad 3Q_2 = 20\lambda \Leftrightarrow Q_2 = \tfrac{20}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$3 + 3Q_1 - 10\lambda = 0 \quad \Leftrightarrow \quad 3Q_1 = -3 + 10\lambda$$

$$\Leftrightarrow \quad Q_1 = \tfrac{-3+10\lambda}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now substitute (4) and (5) in (1):

$$
\begin{aligned}
& 100 - 20\left(\tfrac{-3+10\lambda}{3}\right) - 10\left(\tfrac{20}{3}\lambda\right) = 0 \\
\Leftrightarrow\ & 100 + \tfrac{60}{3} - \tfrac{200\lambda}{3} - \tfrac{200\lambda}{3} = 0 \\
\Leftrightarrow\ & \tfrac{300}{3} + \tfrac{60}{3} - \tfrac{200\lambda}{3} - \tfrac{200\lambda}{3} = 0 \\
\Leftrightarrow\ & \tfrac{360}{3} - \tfrac{400\lambda}{3} = 0 \\
\Leftrightarrow\ & \tfrac{360}{3} = \tfrac{400\lambda}{3} \\
\Leftrightarrow\ & 1200\lambda = 1080 \\
\Leftrightarrow\ & \lambda = \tfrac{9}{10}
\end{aligned}
$$

Substitute (6) in (4):

$$Q_1 = \frac{-3 + 10\left(\frac{9}{10}\right)}{3} = \frac{6}{3} = 2$$

Substitute (6) in (5):

$$Q_2 = \tfrac{20}{3}\left(\tfrac{9}{10}\right) = 6$$

By substituting the value of $Q_1 = 2$, $Q_2 = 6$ (and $\lambda = \frac{9}{10}$) the equation Z will be obtained by the value equation that also function Z is the optimal value function U = 54. The calculation is as follows:

$$Z = 3(2)(6) + 3(6) = 54$$

However, sufficient conditions still need to be met to determine whether the values of Q1 and Q2 are obtained is indeed truly a values (output) which gives the maximum value of 54.

As has been done previously, the second order derivative, in this case can also be expressed in the form of a determinant, is necessary to determine its value as a step to determine sufficient conditions on the results of the optimization process above N = 54.

The function $Z = f(x, y) + \lambda\left(c - g(x, y)\right)$ provisions applicable

d^2z is

$$\begin{Bmatrix} definite\ positive \\ definite\ negative \end{Bmatrix} if\ |\overline{H}| \begin{Bmatrix} < 0 \\ > 0 \end{Bmatrix} \quad with \quad |\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix}$$

$|\overline{H}| =$ Read determinant Hesse-brimmed

f (x, y) and g (x, y) = c are respectively the objective function and constraints.

Finally, let us now investigate whether the value of Z = 54 (UU = 54) were obtained as outputs Q1 and Q2 = 2 = 6 is really have an optimal result. The second order derivative function of Z is

$Z_{11} = 0$ $\quad$ $Z_{12} = 3$; and from $\quad$ $20Q_1 + 10Q_2 = 100$

$Z_{21} = 3$ $\quad$ $Z_{22} = 0$ obtained $\quad$ $g_1 = 20$ dan $g_2 = 10$

$$|\overline{H}| = \begin{vmatrix} 0 & 20 & 10 \\ 20 & 0 & 3 \\ 10 & 3 & 0 \end{vmatrix} = (0 + 600 + 600) - (0 + 0 + 0) = 1200 > 0$$

$|\overline{H}| =$

Because > 0 then 54 is it a truly optimal results (maximum) on the optimization of the above constraints.

Table 3. The Analysis Result (continued)

No	Q_1	Q_2	$2Q_1 + Q_2 = 10$	*U*	*Description*
1	1	6	10	(54)	Optimum
2	2	4	10	48	
3	3	8	10	48	
4	4	2	10	30	

$U = 3Q_1Q_2+3Q_2$

From the table above we obtain information that optimal results can only be obtained through the calculations according to the formulas applicable to optimization with constraints.

6. Conclusion

We have enough to know that mathematics (in this case the derivative or differential equations) have helped us in solving various problems in business and economics in particular to solve the optimum problem. However, it should be understood further that all the problems related to the optimization calculation should first be translated into the language of mathematics or (with another term) made mathematical models of mathematical formulas that can be used to solve the problem satisfactorily.

Especially for solving optimization problems or completion with constraint, we can do with approach and modification of mathematical functions in a linear or non-linear, so that the derivative formulas can be used. Modification of these functions can be done in several ways, among others (Hiller at al, 1994):

1 Substitution Method
2 Lagrange Multiplier Method

7. Suggestions

The problem is formulated and discussed above are merely a common problem in the optimization with constraint, but with non-negative constraints. As for common problems with constraint (which has been translated into the form of the objective function f (x) with g (x) as a function of the constraints), then the necessary and sufficient conditions that must be met are:
1 Keep Terms: Terms Karush-Kuhn-Tucker (KKT) for optimization with constraint.
2 Terms quite: f (x) concave
gi (x) convex (i = 1, 2, ..., m)

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